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Charged-dust distributions in general relativity

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Abstract. The paper presents some simple theorems and relations for charged-dust distributions in general relativity.

1. Introduction

In recent years, the statics and dynamics of charged-dust distributions in general relativity have attracted considerable attention (Bonnor 1965, De 1968, De and Raychaudhuri 1968, Faulkes 1969, Hamoui 1969, Som and Raychaudhuri 1968 a). While some interesting results have emerged, many points remain obscure. In fact, an apparently simple question as to the fate of a collapsing charged-dust distribution for different values of ϵ/ρ (where ϵ is the charge density and ρ is the mass density) and the circumstances under which it may bounce remains largely unanswered.

With a view to throw some light on these problems, the present paper makes an attempt to obtain simple relations and theorems from the Einstein–Maxwell equations for a charged dust without imposing any special symmetry restrictions. Amongst the results that have been obtained the following seem to be of interest:

(i) A formula for the charge density in terms of the electric and magnetic field vectors and the acceleration and vorticity of the dust.

(ii) A theorem that, if the magnetic field vanishes, the electric flux through any element of area bounded by particles of the dust is a constant of motion.

(iii) The result that, in the absence of magnetic field, the vorticity and electric field are orthogonal.

(iv) For an irrotational motion in the absence of magnetic fields, the electric field vector is orthogonal to the surfaces defined by constant values of ϵ/ρ.

(v) A relation between the characteristics of motion (vorticity, acceleration, expansion and shear) and the matter density, the electromagnetic energy density and the Poynting vector (equations (26) and (27)).

(vi) For a dust in irrotational motion in absence of magnetic fields, the expansion (or contraction) cannot be shear-free. This is a generalization of a result obtained earlier De (1968).

(vii) Rainich-like algebraic relations between the $R_{\mu\nu}$ and a theorem that a given $g_{\mu\nu}$ field determines the matter density ρ and the dust velocity vector uniquely and the electromagnetic field tensor up to a duality rotation.

2. The Einstein–Maxwell equations and a proof of the results (i)–(iv)

The Einstein–Maxwell equations for a charged-dust distribution may be written in the form

$$R_{\mu\nu} = -8\pi(T_{\mu\nu} - \tfrac{1}{2}Tg_{\mu\nu}) \tag{1}$$

$$T_{\mu\nu} = \rho v_\mu v_\nu - \frac{1}{4\pi}(F_\mu{}^\alpha F_{\nu\alpha} - \tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}) \tag{2}$$

$$\left.\begin{aligned} F^{\mu\nu}{}_{;\nu} &= 4\pi J^{\mu} \\ {}^{*}F^{\sigma\alpha}{}_{;\alpha} &= \eta^{\sigma\alpha\mu\nu}F_{\mu\nu;\alpha} = 0 \end{aligned}\right\} \tag{3}$$

where the symbols have their usual significance. We now have (Lichnerowicz 1967)

$$T_{\mu\nu} = \rho v_{\mu}v_{\nu} - \frac{1}{4\pi}(\tfrac{1}{2}g_{\mu\nu} - v_{\mu}v_{\nu})(E^2+H^2)$$

$$-\frac{1}{4\pi}(E_{\mu}E_{\nu}+H_{\mu}H_{\nu}) - \frac{1}{4\pi}(v_{\mu}S_{\nu}+v_{\nu}S_{\mu}) \tag{4}$$

$$F_{\alpha\beta} = v_{\alpha}E_{\beta} - v_{\beta}E_{\alpha} - \eta_{\alpha\beta\lambda\mu}v^{\lambda}H^{\mu} \tag{5}$$

where the Poynting vector $\boldsymbol{S}$ is defined as

$$S_{\alpha} = \eta_{\alpha\rho\lambda\mu}E^{\rho}H^{\lambda}v^{\mu}. \tag{6}$$

The current J can be split up into a convection and a conduction part

$$J^{\mu} = \epsilon v^{\mu} + \sigma E^{\mu} \tag{7}$$

where σ is the conductivity of the dust. One obtains from equations (3), (5) and (7)

$$4\pi\epsilon = E^{\alpha}{}_{;\alpha} + E^{\alpha}\dot{v}_{\alpha} - 2H_{\alpha}\omega^{\alpha} \tag{8}$$

where ω^{α} and $\dot{v}^{\alpha}$ are the vorticity and acceleration vectors respectively:

$$\omega^{\beta} = \tfrac{1}{2}\eta^{\beta\alpha\lambda\mu}v_{\alpha;\lambda}v_{\mu} \tag{9}$$

$$\dot{v}_{\mu} = v_{\mu;\alpha}v^{\alpha}. \tag{10}$$

The relation (8) is the one referred to in (i) in the introduction (cf. Som and Raychaudhuri 1968 b). Also with $\sigma = 0$ we have, taking the divergence of equation (1) and using equations (3) and (4),

$$\dot{v}_{\mu} = \frac{\epsilon}{\rho}E_{\mu} \tag{11}$$

so that equation (8) may also be written as

$$4\pi\epsilon = E^{\alpha}{}_{;\alpha} - \frac{\epsilon}{\rho}E^2 - 2H_{\alpha}\omega^{\alpha}. \tag{12}$$

In the case where the electric field vanishes (force-free geodetic motion) the above gives

$$4\pi\epsilon = -2H_{\alpha}\omega^{\alpha}. \tag{13}$$

Examples of the above relation occur in the solutions found by Som and Raychaudhuri (1968 a) and Ellis and Stewart (1968).

If the conductivity of the dust is negligible, then from equations (7), (5) and (3), we obtain after a little reduction

$$2E_{\alpha}\phi^{\alpha\beta} - \tfrac{1}{3}\theta E^{\beta} - v^{\alpha}(E^{\beta}{}_{;\alpha} - E_{\alpha}{}^{;\beta}) - \eta^{\lambda\alpha\beta\sigma}\dot{v}_{\lambda}v_{\alpha}H_{\sigma} - \eta^{\alpha\beta\lambda\sigma}v_{\lambda}H_{\sigma;\alpha} = 0 \tag{14}$$

where the shear $\phi_{\alpha\beta}$ and expansion θ are defined by

$$\phi_{\alpha\beta} = \tfrac{1}{2}(v_{\alpha;\beta}+v_{\beta;\alpha}) - \tfrac{1}{3}(g_{\alpha\beta} - v_{\alpha}v_{\beta})\theta - \tfrac{1}{2}(v_{\alpha}\dot{v}_{\beta} + \dot{v}_{\alpha}v_{\beta}) \tag{15}$$

$$\theta = v^{\alpha}{}_{;\alpha}. \tag{16}$$

The other half of Maxwell's equations may be similarly split up:

$$H^{\mu}\dot{v}_{\mu}+H^{\alpha}{}_{;\alpha}+2E_{\alpha}\omega^{\alpha} = 0 \tag{17}$$

$$2H^{\alpha}\phi_{\alpha}{}^{\mu}-\tfrac{1}{3}\theta H^{\mu}-v^{\alpha}(H^{\mu}{}_{;\alpha}-H_{\alpha}{}^{;\mu})+\eta^{\lambda\alpha\mu\sigma}\dot{v}_{\lambda}v_{\alpha}E_{\sigma}+\eta^{\alpha\mu\lambda\sigma}v_{\lambda}E_{\sigma;\alpha} = 0. \tag{18}$$

Equation (14) may be integrated in the case where the magnetic field vanishes. Let us take a comoving coordinate system so that $v^{\mu} = \delta_0{}^{\mu}$ and the line element has the form

$$\mathrm{d}z^2 = \mathrm{d}x^{0^2}+2g_{0i}\,\mathrm{d}x^0\,\mathrm{d}x^i+g_{ik}\,\mathrm{d}x^i\,\mathrm{d}x^k \tag{19}$$

where we have adopted the convention that the greek indices run from 0 to 3 and the latin indices from 1 to 3. Equation (14) now gives, with $H = 0$,

$$E^i(h_{ik,0}-\tfrac{2}{3}h_{ik}\theta)-\tfrac{1}{3}E_k\theta-E_{k,0} = 0 \tag{20}$$

where

$$\left.\begin{aligned} h_{ik} &\equiv g_{ik}-g_{i0}g_{k0} \\ G^3 &= \sqrt{-g} = \sqrt{-h} \end{aligned}\right\}. \tag{21}$$

Equation (21) yields on integration

$$E^iG^3 = A^i(x^k). \tag{22}$$

Equation (22) leads to the result that the magnitude of the electric field vector is given by

$$-E^2 = \frac{h_{ik}A^iA^k}{G^6}$$

so that the magnitude of the field varies inversely as the area of an element of fluid orthogonal to the direction of the field or, in other words, the flux through an area bounded by the particles of the fluid remains constant.

Again equations (17) and (18) yield for $H = 0$

$$E_i\omega^i = 0 \tag{23}$$

$$\frac{\rho}{\epsilon}\eta^{\alpha\lambda\mu\sigma}g_{0\mu}(g_{0\lambda,\sigma}-g_{0\sigma,\lambda})_{,0}+\eta^{\alpha\lambda\mu\sigma}g_{0\mu}\left\{\dot{g}_{\lambda0}\left(\frac{\rho}{\epsilon}\right)_{,\sigma}-\dot{g}_{\sigma0}\left(\frac{\rho}{\epsilon}\right)_{,\lambda}\right\} = 0. \tag{24}$$

Equation (24) yields, after a little manipulation using equation (23),

$$\frac{\rho}{\epsilon}(\omega^{\alpha}\sqrt{-g})_{,0}+2\frac{\epsilon}{\rho}\eta^{\alpha\mu\lambda\sigma}g_{0\mu}E_{\lambda}\left(\frac{\rho}{\epsilon}\right)_{,\sigma} = 0. \tag{25}$$

Equation (25) leads to three interesting special cases:

(*a*) The motion is irrotational and ϵ/ρ is constant. Equation (25) is trivially satisfied.

(*b*) ϵ/ρ is constant but vorticity exists. Equation (25) gives the result that vorticity varies inversely as the area of an element orthogonal to its direction.

(*c*) The motion is irrotational but ϵ/ρ is not constant. Equation (25) now shows that the electric field is orthogonal to the surfaces defined by constant ϵ/ρ.

3. The expansion equations

We now present two relations between the characteristics of the motion and the energy density and the Poynting vector.

From the identity

$$v^{\mu}{}_{;\mu\alpha} - v^{\mu}{}_{;\alpha\mu} = R_{\sigma\alpha} v^{\sigma}$$

we obtain, using equations (1) and (4),

$$(4\pi\rho + E^2)\left(1 - \frac{\epsilon^2}{\rho^2}\right) + H^2 = 2\frac{\epsilon}{\rho} H^{\alpha}\omega_{\alpha} - 2(\phi^2 - \omega^2) - \frac{\theta^2}{3} - \theta_{,\alpha} v^{\alpha} - \left(\frac{\epsilon}{\rho}\right)_{,\alpha} E^{\alpha} \tag{26}$$

$$2S^{\gamma} = \tfrac{2}{3}\theta_{,\sigma}(g^{\gamma\sigma} - v^{\gamma} v^{\sigma}) - \phi^{\gamma\alpha}{}_{;\alpha} - \phi^{\gamma\alpha}\dot{v}_{\alpha} - 2\phi^2 v^{\gamma} - \eta^{\mu\nu\beta\gamma}(\omega_{\mu;\beta} v_{\nu} - 2\omega_{\mu} v_{\nu}\dot{v}_{\beta}). \tag{27}$$

4. The impossibility of isotropic expansion

We shall consider a charged dust in irrotational motion and shall assume that the field is purely electric, i.e. $\eta^{\sigma\mu\nu\alpha} F_{\mu\nu} v_{\alpha} = 0$. We shall show that under these circumstances the expansion cannot be shear-free.

Now, for an irrotational motion the velocity vector is hypersurface orthogonal and we may introduce a coordinate system in which the world lines of the dust particles will be the t lines and the orthogonal hypersurfaces the three spaces. The line element is now of the form

$$dz^2 = g_{00}\, dt^2 + h_{ik}\, dx^i\, dx^k. \tag{28}$$

With the above line element we obtain from equation (27), for a shear-free expansion with vanishing Poynting vector,

$$\theta = \alpha(t) \tag{29}$$

$$\frac{\dot{G}}{G} = \frac{\alpha}{3}\sqrt{g_{00}}. \tag{30}$$

The condition $\phi_{ik} = 0$ yields

$$h_{ik} = G^2(x^i, t)\psi_{ik}(x^i) \tag{31}$$

with

$$\det|\psi_{ik}| = 1. \tag{32}$$

Also, equation (11) gives

$$(\ln\sqrt{g_{00}})_{,i} = \frac{\epsilon}{\rho} E_i = f_i \frac{x^k}{G} \tag{33}$$

where $f_i = (\epsilon/\rho)\psi_{ik} A^k$ and is a function of the spatial coordinates alone. The consistency of equation (33) would require

$$\eta^{ikl} E_i E_{k,l} = 0 \text{ or } \eta^{ikl} f_i f_{k,l} = 0. \tag{34}$$

Combining equations (30) and (33) we obtain

$$G = X + YT_1 \tag{35}$$

where X and Y are functions of the spatial coordinates alone having the relation

$$\left(\frac{X}{Y}\right)_{,i} = -\frac{f_i}{Y} \tag{36}$$

and T_1 is a function of t alone. If we now substitute from equations (31) and (35) in equation (26) we obtain

$$T_3 + T_4 H = \frac{A}{H^3} + \frac{B}{H^4} \tag{37}$$

where

$$T_2 = \frac{\dot{T}_1}{\alpha}, \qquad T_3 = -\frac{\alpha^2}{3}, \qquad T_4 = -\frac{\dot{\alpha}}{3T_2}, \qquad H = \frac{X}{Y} + T_1.$$

$$A = 4\pi\rho\left(1 - \frac{\epsilon^2}{\rho^2}\right)H^3 + \left(\frac{\epsilon}{\rho}\right)_{,i} E^i H^3$$

$$B = E^2\left(1 - \frac{\epsilon^2}{\rho^2}\right)H^4.$$

Thus A and B are functions of the spatial coordinates alone. It is not difficult to see that equation (37) leads to the condition that either T_1 and T_2 are constants or that E vanishes, i.e. the electric field and along with it the charge density would vanish if the spatial expansion were shear-free and non-vanishing.

5. The algebraic relation between the $R_{\mu\nu}$

As is well known, Rainich (1925) and later Misner and Wheeler (1957) gave some relations between the $R_{\mu\nu}$ that are necessary and sufficient for the $R_{\mu\nu}$ to represent a pure electromagnetic field. In our case there will be similar relations but somewhat modified owing to the presence of the dust field. We may write from equations (1) and (2)

$$\begin{aligned} S_{\mu\nu} &\equiv R_{\mu\nu} + 8\pi(\rho v_\mu v_\nu - \tfrac{1}{2}\rho g_{\mu\nu}) \\ &= 2(F_{\mu\alpha}F_\nu{}^\alpha - \tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}) \end{aligned} \tag{38}$$

and hence the Rainich algebraic relations are

$$S^\alpha{}_\alpha = 0 \tag{39}$$

$$S_{\alpha\beta}S^{\alpha\sigma} = \tfrac{1}{4}\delta_\beta{}^\sigma(S_{\alpha\mu}S^{\alpha\mu}). \tag{40}$$

Or, in terms of $R_{\mu\nu}$,

$$\left.\begin{aligned} R &= 8\pi\rho > 0. \\ R_{\alpha\beta}R^{\alpha\sigma} - RR^\sigma{}_\beta + \tfrac{1}{4}R^2\delta^\beta{}_\sigma + RR_{\alpha\beta}v^\alpha v^\sigma + RR^\sigma{}_\alpha v^\alpha v_\beta & \\ = \tfrac{1}{4}\delta^\sigma{}_\beta(R_{\mu\nu}R^{\mu\nu} + 2RR_{\mu\nu}v^\mu v^\nu) & \end{aligned}\right\}. \tag{41}$$

Besides the $R_{\mu\nu}$ the above relations involve the velocity vector v^μ. Let us investigate whether a given $R_{\mu\nu}$ satisfying equation (41) determines ρ, v^μ uniquely. In that case the electromagnetic field tensor would be determined up to a duality rotation. In view of equation (41) ρ is determined uniquely. If possible, let v^μ, u^μ be two time-like unit vectors satisfying equation (41) with $v^\mu \neq u^\mu$. Then we would have

$$R_{\alpha\beta}v^\alpha v^\sigma + R^\sigma{}_\alpha v^\alpha v_\beta = R_{\alpha\beta}u^\alpha u^\sigma + R^\sigma{}_\alpha u^\alpha u_\beta. \tag{42}$$

If we now take a locally Lorentz frame such that $v^\mu = \delta_0{}^\mu$ and u^μ has only the components u^0 and u^1 non-vanishing, then, from equation (42) with $\beta = \sigma = 0$, we have

$$R_{00}(1 - u^{0^2}) = R_{01}u^1u^0. \tag{43}$$

Also for $\beta = 1$, $\sigma = 0$

$$0 = R_{11}u^1u^0 - R_{00}u^0u^1. \tag{44}$$

Equation (44) gives, in view of equations (2) and (4),

$$R_{00} = R_{11} \rightarrow \rho = -(E_1{}^2 + H_1{}^2)$$

which contradicts the physical requirement $\rho > 0$. The case $\rho = 0$ is already known to lead to unique values of $F_{\mu\nu}$ up to a duality rotation.

We shall not attempt here to write out differential relations with the $R_{\mu\nu}$ to replace the Maxwell equations in the manner done in the already unified theory. Such a replacement seems possible but the equations would be so complicated as to be of hardly any interest for the present discussion.

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